Problem Set 1

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 1 covers materials from chapter 0 and chapter 1.

1. Consider the closed intervals $A = [0, 4]$ and $B = [3, 6]$. Express each of the following sets as a union of disjoint intervals.
   (a) $\mathbb{R} \setminus A$
   (b) $\mathbb{R} \setminus (A \cap B)$
   (c) $\mathbb{R} \setminus \big((A \cup B) \setminus (A \cap B)\big)$
   (d) $A \cup \big(B \cap (\mathbb{R} \setminus A)\big)$
2. Solve each of the following inequalities. Write each final answer in the interval notation.
   (a) $\dfrac{(x-1)(x+2)}{(x+1)(x-2)} \geq 0$
   (b) $\dfrac{x^2+1}{(x+1)(x-2)} \geq 0$
   (c) $x - \dfrac{3}{x} \geq 2$
   (d) $\dfrac{4-x}{(1-x)^{2/3}(2-x)^{1/3}} > 0$
   (e) $3 < |2x - 1| \leq 5$
   (f) $|x^2 - 4x - 1| > 4$
3. Evaluate each of the following sums.
   (a) $30 + 32 + 34 + 36 + \cdots + 50$
   (b) $1 \cdot 4 \cdot 7 + 2 \cdot 5 \cdot 8 + 3 \cdot 6 \cdot 9 + \cdots + 30 \cdot 33 \cdot 36$
   (c) $1 + 1.01 + 1.01^2 + 1.01^3 + \cdots + 1.01^{10}$
4. Let n be a positive integer. Evaluate each of the following sums in terms of n.
   (a) $\displaystyle\sum_{k=1}^{n} (k+2)(k-3)$
   (b) $1 + 5^{\frac{1}{3}} + 5^{\frac{2}{3}} + 5 + 5^{\frac{4}{3}} + \cdots + 5^n$
5. Find the following functions as required.
   (a) Consider a circular cylinder whose volume is 100 cubic units. Let r and h be the radius and height of the circular cylinder respectively. Express r as a function of h.
   (b) Consider a cube. Let d be the length of a diagonal, l be the length of an edge, A be the surface area and V be the volume of the cube. Express each of l, A and V as a function of d.
   (c) Let P be a point in the first quadrant which lies on the graph of the function
   $$f(x) = \sqrt{x},$$
   let (x, y) be the coordinates of P, and let m be the slope of the line joining P and the origin. Express each of x and y as a function of m.

6. Write down the natural domain of each of the following functions.

(a) $f(x) = \sqrt{\dfrac{x+2}{x+3}}$

(b) $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x+3}}$

(c) $f(x) = \sqrt{\dfrac{x^2-1}{x-2}}$

(d) $f(x) = \sqrt{|x|}$

(e) $f(x) = |\sqrt{x}|$

(f) $f(x) = \dfrac{1}{x-|x|}$

(g) $f(x) = \dfrac{1}{\sin x}$

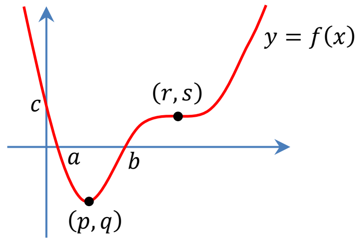
(h) $f(x) = \ln|x|$

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a function whose graph is as shown in the diagram on the right. Find the natural domain and the range of each of the following functions g, h and m:

(a) $g(x) = 1 - f(x)$

(b) $h(x) = \sqrt[4]{f(x)}$

(c) $m(x) = \dfrac{2}{f(x)}$



8. (a) Let a be a real number and consider the function $f: \mathbb{R} \to \mathbb{R}$ defined piecewise by

$$f(x) = \begin{cases} x - a & \text{if } x > a \\ 0 & \text{if } x \le a \end{cases}.$$

Show that for every real number x we have

$$f(x) = \frac{1}{2}(x-a) + \frac{1}{2}|x-a|.$$

(b) Recall the taxi fare function $P: (0, +\infty) \to \mathbb{R}$ in Example 1.28, which is defined by

$$P(x) = \begin{cases} 29 & \text{if } 0 < x \le 2 \\ 10.5x + 8 & \text{if } 2 < x \le 9 \\ 7x + 39.5 & \text{if } x > 9 \end{cases}.$$

Using the result from (a), express this function P as the sum or difference of a linear polynomial and absolute values of linear polynomials, i.e. write P in the form

$$P(x) = (b_0x + c_0) + b_1|x + c_1| + b_2|x + c_2| + \cdots$$

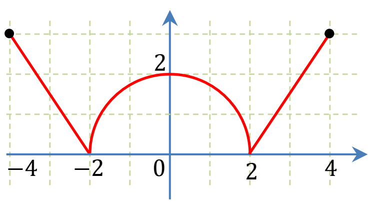
where b_k and c_k are real numbers.

9. For each of the following, find a formula for the function whose graph is the given curve.

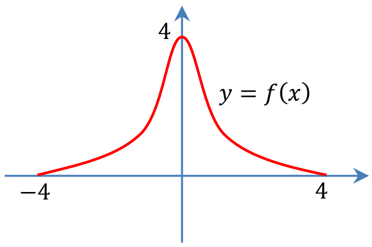
(a) The line segment joining the points $(1, -3)$ and $(5, 7)$.

(b) The bottom half of the parabola with equation $x + (y-1)^2 = 0$.

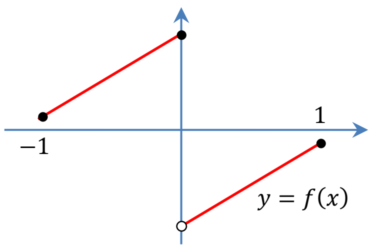
(c)



10. For each of the following functions, determine whether it is an odd function and whether it is an even function:
    (a) $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^5+x^3+x$.
    (b) $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=\sqrt{|x|}$.
    (c) $h:\mathbb{R}\to\mathbb{R}$ defined by $h(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ -1 & \text{if } x \text{ is irrational}\end{cases}$.
11. Show that every function $f:\mathbb{R}\to\mathbb{R}$ can be decomposed as the sum of an odd function and an even function. In other words, construct two functions $f_1, f_2:\mathbb{R}\to\mathbb{R}$ (in terms of f) such that f_1 is odd, f_2 is even and
$$f=f_1+f_2.$$
12. Prove the following properties of odd and even functions.
    (a) Let $f:\mathbb{R}\to\mathbb{R}$ be an odd function and let $g:\mathbb{R}\to\mathbb{R}$ be the function $g(x)=|f(x)|$. Show that g is even.
    (b) Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let $g:\mathbb{R}\to\mathbb{R}$ be the function $g(x)=f(|x|)$. Show that g is even.
    (c) Let $f,g:\mathbb{R}\to\mathbb{R}$ be a pair of functions such that f is even and g is odd. Show that the composition $f\circ g:\mathbb{R}\to\mathbb{R}$ is an even function.
13. Let f be a function whose graph $y=f(x)$ is as follows.



On the same coordinate plane, sketch the graph of
$$y=\frac{1}{3}f(|x|-1)-2.$$
14. Determine whether the following functions are one-to-one:
    (a) $f:\{1,2,3,4,\dots\}\to\mathbb{R}$ defined by $f(n)=$ the n^{th} prime number.
    (b) $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\ln(1+x^2)$.
    (c) $f:(-1,+\infty)\to\mathbb{R}$ defined by $f(x)=\dfrac{x}{\sqrt{x+1}}$.
    (d) $f:[-1,1]\to\mathbb{R}$ whose graph is as follows:



15. For each of the following functions, determine whether it has an inverse or not. Find its inverse if it has one.
    (a) $f\colon \mathbb{R}\setminus\{-2\} \to \mathbb{R}$ defined by $f(x) = \dfrac{2-x}{2+x}$
    (b) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{e^x}{1+e^x}$
    (c) $f\colon (\pi/2, 3\pi/2) \to \mathbb{R}$ defined by $f(x) = \tan x$
    (d) $f\colon \left(-\sqrt{\pi/2}, \sqrt{\pi/2}\right) \to \mathbb{R}$ defined by $f(x) = \tan(x^2)$
16. Solve each of the following equations as instructed.
    (a) Find all the real numbers x which satisfy the the equation
    $$\cos 2x + 3\sin x - 2 = 0.$$
    (b) Find all the numbers $x \in [0, 4\pi]$ which satisfy the equation
    $$\cos\left(x - \frac{\pi}{3}\right) + 2\cos\left(x + \frac{\pi}{3}\right) = 0.$$
17. Prove the following trigonometric identities:
    (a) $\dfrac{\sec(-x) + \sin\left(-\frac{\pi}{2} - x\right)}{\csc(3\pi - x) - \cos\left(\frac{3\pi}{2} + x\right)} = \tan^3 x$
    (b) $\dfrac{\cos^2 x - \sin x\cos x + \tan x}{\cos^2 x + \sin x\cos x - \tan x} = \dfrac{1 + \tan^3 x}{1 - \tan^3 x}$
    (c) $\dfrac{\sin 3x}{\sin x} + \dfrac{\cos 3x}{\cos x} = 4\cos 2x$
    (d) $\dfrac{\sin x}{1 + \sin x} = 1 - \sec^2 x + \sec x\tan x$
18. Let $x \in (-\pi, \pi)$ and denote $t = \tan(x/2)$. Show that
    $$\cos x = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad \sin x = \frac{2t}{1+t^2}.$$
19. Without using calculators, find the value of each of the following:
    (a) $\arccos\left(\sin\dfrac{5\pi}{3}\right)$ (b) $\sec(\arctan 1)$ (c) $\arccos(\cos 10)$
20. The ***hyperbolic functions*** sinh, cosh and tanh are defined using the natural exponential function as
    $$\sinh x \coloneqq \frac{e^x - e^{-x}}{2}, \qquad \cosh x \coloneqq \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \tanh x \coloneqq \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
    respectively.
    (a) For each of the functions sinh, cosh and tanh, find its natural domain and its range.
    (b) For each of the functions sinh, cosh and tanh, determine whether it is odd and whether it is even.